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On the Relation Between Orthogonal, Symplectic and Unitary Matrix Ensembles

Harold Widom[1]

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For the unitary ensembles of $N \times N$ Hermitian matrices associated with a weight function w there is a kernel, expressible in terms of the polynomials orthogonal with respect to the weight function, which plays an important role. For the orthogonal and symplectic ensembles of Hermitian matrices there are 2×2 matrix kernels, usually constructed using skew-orthogonal polynomials, which play an analogous role. These matrix kernels are determined by their upper left-hand entries. We derive formulas expressing these entries in terms of the scalar kernel for the corresponding unitary ensembles. We also show that whenever w'/w is a rational function the entries are equal to the scalar kernel plus some extra terms whose number equals the order of w'/w. General formulas are obtained for these extra terms. We do not use skew-orthogonal polynomials in the derivations.

KEY WORDS: Random matrices; matrix kernels; unitary ensembles; orthogonal ensembles; symplectic ensembles; Laguerre ensembles.

1. INTRODUCTION

In the most common ensembles of $N \times N$ Hermitian matrices the probability density $P_N(x_1, ..., x_N)$ that the eigenvalues lie in infinitesimal neighborhoods of $x_1, ..., x_N$ is given by

$$P_N(x_1, ..., x_N) = c_N \prod_{j<k} |x_j - x_k|^\beta \prod_j w(x_j)$$

where $\beta = 1, 2$ or 4 (corresponding to the orthogonal, unitary and symplectic ensembles, respectively), $w(x)$ is a weight function and c_N is a normalization constant.

[1] Department of Mathematics, University of California, Santa Cruz, California 95064; e-mail address: widom@math.ucsc.edu.

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For the unitary matrix ensembles an important role is played by the kernel

$$K_N(x, y) = \sum_{k=0}^{N-1} \varphi_k(x)\, \varphi_k(y) \tag{1.1}$$

where $\{\varphi_k(x)\}$ is the sequence obtained by orthonormalizing the sequence $\{x^k w(x)^{1/2}\}$. The probability density is expressed in terms of it by

$$P_N(x_1, ..., x_N) = \frac{1}{N!} \det(K_N(x_j, x_k))_{j, k=1, ..., N}.$$

More generally the n-point correlation function $R_n(x_1, ..., x_n)$, the probability density that n of the eigenvalues, irrespective of order, lie in infinitesimal neighborhoods of $x_1, ..., x_n$, is given by the formula

$$R_n(x_1, ..., x_n) = \det(K_N(x_j, x_k))_{j, k=1, ..., n}. \tag{1.2}$$

And the probability $E(0; J)$ that the set J contains no eigenvalues is equal to the Fredholm determinant of the kernel $K_N(x, y)\, \chi_J(y)$, where χ denotes characteristic function.

For the orthogonal and symplectic ensembles there are 2×2 matrix kernels which play analogous roles. In this case the determinant in (1.2) is to be interpreted as a quaternion determinant (it is a linear combination of products of traces of products of matrix entries of the block matrix on the right side), and the square of $E(0; J)$ equals the Fredholm determinant of the matrix kernel. (The last fact can be deduced from the computation in ref. 3, Section A.7. A direct derivation is given in ref. 10.) In the case of the orthogonal ensembles we shall always assume that N is even. The kernels for the orthogonal and symplectic ensembles are of the form

$$K_{N1}(x, y) = \begin{pmatrix} S_{N1}(x, y) & S_{N1}D(x, y) \\ IS_{N1}(x, y) - \varepsilon(x - y) & S_{N1}(y, x) \end{pmatrix} \tag{1.3}$$

and

$$K_{N4}(x, y) = \frac{1}{2} \begin{pmatrix} S_{N4}(x, y) & S_{N4}D(x, y) \\ IS_{N4}(x, y) & S_{N4}(y, x) \end{pmatrix} \tag{1.4}$$

respectively. Here $\varepsilon(x) = \frac{1}{2} \operatorname{sgn}(x)$ and the explanation for the notation is this: the $S_{N\beta}(x, y)$ are certain sums of products and if $S_{N\beta}$ is the operator with kernel $S_{N\beta}(x, y)$ then $S_{N\beta}D(x, y)$ is the kernel of $S_{N\beta}D$ ($D =$ differentiation) and $IS_{N\beta}(x, y)$ is the kernel of $IS_{N\beta}$ ($I=$ integration, more or less). We shall write these out below. One can see from this description that once the kernels $S_{N\beta}(x, y)$ are known then so are the others.

Matrix kernels were first introduced by Dyson[1] for his circular ensembles and he established the analogue of formula (1.2) for the correlation functions. Later, Mehta[3] and Mehta and Mahoux[2] found matrix kernels for the ensembles of Hermitian matrices, and expressed them in terms of systems of skew-orthogonsal polynomials. These are like orthogonal polynomials but the inner product (different in the $\beta=1$ and $\beta=4$ cases) is antisymmetric instead of symmetric. In terms of them one obtains for $S_{N\beta}(x, y)$ sums like the one in Eq. (1.1) but which are a little more complicated. A problem here is that the skew-orthogonal polynomials are not always that easy to compute and, even if they are, the sums involving them may not be easy to handle. For example, one is often interested in scaling limits as $N\to\infty$ and in order to do this it helps to have a good representation for the sum.

In this paper we shall not use skew-orthogonal polynomials at all. Instead, we shall use the expressions for the various matrix kernels in the general form given in ref. 10, and derive general formulas for the $S_{N\beta}(x, y)$ in terms of the scalar kernel $K_N(x, y)$ given by Eq. (1.1), with N replaced by $2N$ when $\beta=4$ and w replaced by w^2 when $\beta=1$. More exactly, we shall express the operators whose kernels are the $S_{N\beta}(x, y)$ in terms of the operator whose kernel is $K_N(x, y)$. These are given in Theorem 1 below.

The formulas can be brought to a very concrete form whenever the support $\mathscr{D}$ of w is a finite union of finite or infinite intervals and w'/w is equal to a rational function on $\mathscr{D}$. (Such weight functions are called *semi-classical* since they include the weight functions for all the classical orthogonal polynomials.) We find then that the $S_{N\beta}(x, y)$ are equal to the appropriate scalar kernel $K_N(x, y)$ plus some extra terms whose number is independent of N. This number equals the order of w'/w, the sum of the orders of its poles in the extended complex plane. We must also count as a simple pole any end-point of $\mathscr{D}$ where w'/w is analytic. Thus, for the Gaussian ensembles ($w(x)=e^{-x^2}$) and Laguerre ensembles ($w(x)=x^\alpha e^{-x}$) there will one extra term because of the simple poles at ∞ and 0 respectively, and for the Jacobi ensemble ($w(x)=(1-x)^\alpha(1+x)^\beta$) there will be two extra terms because of the simple poles at ± 1. For the Legendre ensemble on $(-1, 1)$ there will also be two extra terms although $w'/w=0$ in this case.

We shall produce explicit formulas for the extra terms, which are given in Theorem 2. These will be used to work out the cases of the Gaussian ensembles (well-known[4]) and the Laguerre ensembles (known apparently only in the case $\alpha=0$[6, 7]).

To apply our formulas to the Laguerre ensemble we require at first that $\alpha > 0$ so that Theorem 1 is applicable. The formulas for general $\alpha > -1$ are then obtained by analytic continuation. Similar analytic continuation arguments apply quite generally. (See the remark at the end of Section 3.) For example, for the Legendre ensemble we would start with the formulas for the Jacobi ensemble for $\alpha, \beta > 0$ and then take the analytic continuation (or limit) to obtain the fomulas for $\alpha = \beta = 0$. This is the reason the end-points ± 1 count as poles.

The recent announcement(8) has some elements in common with ours. A generalization of the Laguerre ensemble was considered there where e^{-x} was replaced by the exponential of an arbitrary polynomial and the occurrence of only finitely many extra terms was established, without their being evaluated, using skew-orthogonal polynomials. This fact was used to deduce universality for this class of ensembles.

2. THE GENERAL IDENTITIES

We start with the expressions for the various matrix kernels in the form given in ref. 10. (The notation here is slightly different.) Taking the symplectic ensembles first, we let $\{p_j(x)\}$ be any sequence of polynomials of exact degree j and define $\varphi_j(x) = p_j(x)\, w(x)^{1/2}$. Let M be the $2N \times 2N$ matrix with j, k entry $(j, k = 0, ..., 2N-1)$

$$m_{jk} = \tfrac{1}{2} \int (p_j(x)\, p_k'(x) - p_j'(x)\, p_k(x))\, w(x)\, dx$$

$$= \tfrac{1}{2} \int (\varphi_j(x)\, \varphi_k'(x) - \varphi_j'(x)\, \varphi_k(x))\, dx. \tag{2.1}$$

This matrix is invertible and we write $M^{-1} = (\mu_{jk})$. Then

$$S_{N4}(x, y) = \sum_{j,k=0}^{2N-1} \varphi_j'(x)\, \mu_{jk}\, \varphi_k(y) \tag{2.2}$$

and

$$IS_{N4}(x, y) = \sum_{j,k=0}^{2N-1} \varphi_j(x)\, \mu_{jk}\, \varphi_k(y), \qquad S_{N4} D(x, y) = -\sum_{j,k=0}^{2N-1} \varphi_j'(x)\, \mu_{jk}\, \varphi_k'(y).$$

Any family of polynomials leads to the same matrix kernel. Of course at this point the formulas look quite bad because of the μ_{jk}.

For the orthogonal ensembles we take the p_j as before but this time define $\varphi_j(x)=p_j(x)\,w(x)$ and let M be the $N\times N$ matrix with j, k entry $(j, k=0, ..., N-1)$

$$m_{jk}=\iint \varepsilon(x-y)\,p_j(x)\,p_k(y)\,w(x)\,w(y)\,dy\,dx=\int \varphi_j(x)\,\varepsilon\varphi_k(x)\,dx. \qquad (2.3)$$

Here ε denotes the operator with kernel $\varepsilon(x-y)$. Again M is invertible, we write $M^{-1}=(\mu_{jk})$, and the formulas for the kernels are

$$S_{N1}(x, y)=-\sum_{j,k=0}^{N-1} \varphi_j(x)\,\mu_{jk}\,\varepsilon\varphi_k(y),$$

$$IS_{N1}(x, y)=-\sum_{j,k=0}^{N-1} \varepsilon\varphi_j(x)\,\mu_{jk}\,\varepsilon\varphi_k(y),$$

$$S_{N1}D(x, y)=\sum_{j,k=0}^{N-1} \varphi_j(x)\,\mu_{jk}\,\varphi_k(y).$$

We change notation so that we can treat the two cases at the same time—we shall see that they are interrelated. We continue to use the notations N and w, but when $\beta=4$ the N here will be the $2N$ of Eq. (2.2) and when $\beta=1$ the w here will be square of the weight function in Eq. (2.3). Thus in both cases N is even, we take p_j to be polynomials of exact degree j and set $\varphi_j=p_j\,w^{1/2}$. The matrices $(m_{jk}^{(\beta)})$ and $(\mu_{jk}^{(\beta)})$ are the M and M^{-1} corresponding to the $\beta=4$ and 1 ensembles. We set

$$S_N^{(4)}(x, y)=\sum_{j,k=0}^{N-1} \varphi_j'(x)\,\mu_{jk}^{(4)}\,\varphi_k(y), \qquad S_N^{(1)}(x, y)=-\sum_{j,k=0}^{N-1} \varphi_j(x)\,\mu_{jk}^{(1)}\,\varepsilon\varphi_k(y).$$

Finally, $K_N(x, y)$ will denote the $\beta=2$ scalar kernel (1.1).

We denote by $\mathscr{H}$ the linear space spanned by the functions $\varphi_0, ..., \varphi_{N-1}$, in other words the set of all functions of the form $w^{1/2}$ times a polynomial of degree less than N. We denote by K be the projection operator onto $\mathscr{H}$. Its kernel is $K_N(x, y)$. Finally, we denote by $S^{(4)}$ the operator with kernel $S_N^{(4)}(x, y)$ and by $S^{(1)\prime}$ the operator with kernel $S_N^{(1)}(y, x)$.

The following lemma will identify these operators. We think of our weight functions as defined on all of $\mathbf{R}$, and our basic assumptions are

$$\mathscr{H}\subset L^1(\mathbf{R}), \qquad D\mathscr{H}\subset L^1(\mathbf{R}). \qquad (2.4)$$

The former is needed even to define the ensembles. The latter is restrictive and implies in particular that all the φ_k are absolutely continuous. (If our weight function is initially defined on a domain $\mathscr{D}$ it must vanish at the end-points of $\mathscr{D}$ for its extension to all of $\mathbf{R}$, defined by setting it equal to zero outside $\mathscr{D}$, to be absolutely continuous.) We use the notations $D_{\mathscr{H}}$ and $\varepsilon_{\mathscr{H}}$ for the restrictions of the operators D and ε, respectively, to $\mathscr{H}$.

Lemma. The operators $KD_{\mathscr{H}}$ and $K\varepsilon_{\mathscr{H}}$ are invertible and

$$S^{(4)}|_{\mathscr{H}} = D(KD_{\mathscr{H}})^{-1}, \qquad S^{(4)}|_{\mathscr{H}^\perp} = 0,$$

$$S^{(1)\prime}|_{\mathscr{H}} = \varepsilon(K\varepsilon_{\mathscr{H}})^{-1}, \qquad S^{(1)\prime}|_{\mathscr{H}^\perp} = 0.$$

Proof. Integrating by parts the second integral in Eq. (2.1) shows that $m_{jk}^{(4)} = \int \varphi_j(x)\, \varphi_k'(x)\, dx$. (This is where the absolute continuity of the φ_k come in.) Thus for $i = 0,\ldots, N-1$,

$$S^{(4)}K\varphi_i' = \sum_{j,\,k} \varphi_j' \mu_{jk}^{(4)}(\varphi_k, \varphi_i') = \sum_{j,\,k} \varphi_j' \mu_{jk}^{(4)} m_{ki}^{(4)} = \sum_j \varphi_j' \delta_{ji} = \varphi_i'.$$

Since the φ_i span $\mathscr{H}$ we see that $S^{(4)}K\varphi' = \varphi'$ for all $\varphi \in \mathscr{H}$. This shows that $KD_{\mathscr{H}}$ is a one-one, and hence invertible, operator on $\mathscr{H}$, and also that $S^{(4)}|_{\mathscr{H}} = D(KD_{\mathscr{H}})^{-1}$. Of course $S^{(4)}|_{\mathscr{H}^\perp} = 0$ since each $\varphi_k \in \mathscr{H}$. This proves the first part of the lemma. For the second, observe that by the antisymmetry of $(m_{jk}^{(1)})$ the formula for $S_N^{(1)}(y, x)$ can be obtained from the formula for $S_N^{(4)}(x, y)$ by replacing $m_{jk}^{(4)}$ by $m_{jk}^{(1)}$ and $\varphi_j'(x)$ by $\varepsilon\varphi_j(x)$. Thus the second part of the lemma can be proved just as the first, replacing D everywhere by ε.

To identify $(KD_{\mathscr{H}})^{-1}$ and $(K\varepsilon_{\mathscr{H}})^{-1}$ more concretely we shall enlarge the domains of D and ε. We have $D\mathscr{H} \subset L^1(\mathbf{R})$ by assumption, and $\varepsilon\mathscr{H} \subset L^\infty(\mathbf{R})$ since $\mathscr{H} \subset L^1(\mathbf{R})$. It is easy to see that the operators

$$D\colon \mathscr{H} + \varepsilon\mathscr{H} \to \mathscr{H} + D\mathscr{H}, \qquad \varepsilon\colon \mathscr{H} + D\mathscr{H} \to \mathscr{H} + \varepsilon\mathscr{H}$$

are mutual inverses. In the following, $I_{\mathscr{H}+D\mathscr{H}}$ and $I_{\mathscr{H}+\varepsilon\mathscr{H}}$ will denote the identity operators on the spaces $\mathscr{H} + D\mathscr{H}$ and $\mathscr{H} + \varepsilon\mathscr{H}$, respectively.

Theorem 1. We have

$$S^{(4)} = (I_{\mathscr{H}+D\mathscr{H}} - (I-K)\, DK\varepsilon)^{-1}\, K, \tag{2.5}$$

$$S^{(1)\prime} = (I_{\mathscr{H}+\varepsilon\mathscr{H}} - (I-K)\, \varepsilon KD)^{-1} K. \tag{2.6}$$

Proof. Since D and ε are mutual inverses we might guess that a good approximation to the inverse of $KD_{\mathscr{H}}$ is $K\varepsilon_{\mathscr{H}}$, where $\varepsilon_{\mathscr{H}}$ denotes the restriction of ε to $\mathscr{H}$. With this in view, we compute

$$K\varepsilon KD_{\mathscr{H}} = K\varepsilon D_{\mathscr{H}} - K\varepsilon(I-K)\,D_{\mathscr{H}} = I_{\mathscr{H}} - K\varepsilon(I-K)\,D_{\mathscr{H}}$$

where $I_{\mathscr{H}}$ denotes the identity operator on $\mathscr{H}$. The operator on the right side is invertible since both $KD_{\mathscr{H}}$ and $K\varepsilon_{\mathscr{H}}$ are, and we deduce that

$$(KD_{\mathscr{H}})^{-1} = (I_{\mathscr{H}} - K\varepsilon(I-K)\,D_{\mathscr{H}})^{-1}\,K\varepsilon_{\mathscr{H}}.$$

Hence by the lemma,

$$\begin{aligned} S^{(4)}|_{\mathscr{H}} &= D_{\mathscr{H}}(KD_{\mathscr{H}})^{-1} = KD_{\mathscr{H}}(KD_{\mathscr{H}})^{-1} + (I-K)\,D_{\mathscr{H}}(KD_{\mathscr{H}})^{-1} \\ &= I_{\mathscr{H}} + (I-K)\,D_{\mathscr{H}}(I_{\mathscr{H}} - K\varepsilon(I-K)\,D_{\mathscr{H}})^{-1}\,K\varepsilon_{\mathscr{H}}. \end{aligned}$$

Recall that the domain of ε is $\mathscr{H} + D\mathscr{H}$ and set

$$A = (I-K)\,D_{\mathscr{H}} \colon \mathscr{H} \to \mathscr{H} + D\mathscr{H}, \qquad B = K\varepsilon \colon \mathscr{H} + D\mathscr{H} \to \mathscr{H}.$$

Then

$$I_{\mathscr{H}+D\mathscr{H}} + (I-K)\,D_{\mathscr{H}}(I_{\mathscr{H}} - K\varepsilon(I-K)\,D_{\mathscr{H}})^{-1}\,K\varepsilon$$

is equal in this notation to $I_{\mathscr{H}+D\mathscr{H}} + A(I_{\mathscr{H}} - BA)^{-1}\,B$. This in turn equals $(I_{\mathscr{H}+D\mathscr{H}} - AB)^{-1}$. Hence restricting to $\mathscr{H}$ gives

$$S^{(4)}|_{\mathscr{H}} = (I_{\mathscr{H}+D\mathscr{H}} - (I-K)\,DK\varepsilon)^{-1}|_{\mathscr{H}}.$$

Since $S^{(4)}|_{\mathscr{H}^{\perp}} = K|_{\mathscr{H}^{\perp}} = 0$ this gives (2.5), and (2.6) is obtained by an analogous argument, interchanging the roles of D and ε.

Remark. The identities of the theorem may be restated in the rather more complicated form

$$\begin{aligned} S^{(4)} &= K + (I-K)\,DK\varepsilon(I_{\mathscr{H}+D\mathscr{H}} - (I-K)\,DK\varepsilon)^{-1}\,K, \\ S^{(1)\prime} &= K + (I-K)\,\varepsilon KD(I_{\mathscr{H}+\varepsilon\mathscr{H}} - (I-K)\,\varepsilon KD)^{-1}\,K. \end{aligned}$$

The summands on the right may be thought of as corrections and we see that they will be of finite rank (independent of N) whenever $(I-K)\,DK\varepsilon$ and $(I-K)\,\varepsilon KD$ are. This will be true whenever the commutator $[D, K]$ is, which will be the case in what follows.

3. THE CASE OF RATIONAL w'/w

We assume now that w'/w is a rational function on the support of w and, at first, that Eq. (2.4) holds so that Theorem 1 is applicable. We explain at the end of this section how to remove the restriction in the cases of greatest interest. From now on it will be convenient to take the p_j to be the polynomials orthonormal with respect to the weight function w so that the φ_j are orthonormal with respect to Lebesgue measure.

It follows from the Christoffel-Darboux formula that there is a representation

$$K_N(x, y) = a_N \frac{\varphi_N(x)\,\varphi_{N-1}(y) - \varphi_{N-1}(x)\,\varphi_N(y)}{x-y}$$

$$= a_N(\varphi_N(x)\ \varphi_{N-1}(x)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_N(y) \\ \varphi_{N-1}(y) \end{pmatrix} \Big/ (x-y) \tag{3.1}$$

for a certain constant a_N. This holds for an arbitrary weight function. Whenever w'/w is a rational function there is a differentiation formula

$$\begin{pmatrix} \varphi'_N \\ \varphi'_{N-1} \end{pmatrix} = \begin{pmatrix} A & B \\ -C & -A \end{pmatrix} \begin{pmatrix} \varphi_N \\ \varphi_{N-1} \end{pmatrix}$$

where $A(x)$, $B(x)$ and $C(x)$ are rational functions whose poles are among those of w'/w, counting multiplicity. (See ref. 9, Section 6.) From this and Eq. (3.1) we find that the kernel of $[D, K]$, which equals $(\partial_x + \partial_y)$ $K_N(x, y)$, is equal to

$$a_N(\varphi_N(x)\ \varphi_{N-1}(x)) \begin{pmatrix} \dfrac{C(x)-C(y)}{x-y} & \dfrac{A(x)-A(y)}{x-y} \\[2ex] \dfrac{A(x)-A(y)}{x-y} & \dfrac{B(x)-B(y)}{x-y} \end{pmatrix} \begin{pmatrix} \varphi_N(y) \\ \varphi_{N-1}(y) \end{pmatrix}. \tag{3.2}$$

It follows from this that $[D, K]$ is a finite rank operator, and that its kernel is expressible in terms of the functions

$$x^k \varphi_{N-1}(x), \qquad x^k \varphi_N(x), \qquad (0 \leqslant k < n_\infty) \tag{3.3}$$

where n_∞ is the order of w'/w at infinity and, for each finite pole x_i of w'/w, the functions

$$(x-x_i)^{-k-1}\,\varphi_{N-1}(x), \qquad (x-x_i)^{-k-1}\,\varphi_N(x), \qquad (0 \leqslant k < n_{x_i}) \tag{3.4}$$

where n_{x_i} is the order of w'/w at x_i. This is seen by expanding the functions appearing in the central matrix in (3.2), which is simple algebra.

In the space spanned by these $2n$ functions (n is the total order of w'/w) there is, when $N \geqslant n$, a subspace of dimension n contained in $\mathscr{H}$ and a subspace of dimension n contained in $\mathscr{H}^\perp$. To see the first, an inductive argument using the three-term recurrence formula shows that the subspace spanned by the functions (3.3) contains the n_∞ functions φ_{N-k} $(0 < k \leqslant n_\infty)$ which lie in $\mathscr{H}$. The functions (3.4) span a space of dimension $2 \sum n_{x_i}$ consisting of functions which equal $w^{1/2}$ times rational functions which may have poles at the x_i of order n_{x_i}. A function in this space will belong to $\mathscr{H}$ if the principal parts at all these poles vanish. This gives $\sum n_{x_i}$ conditions in a space of dimension $2 \sum n_{x_i}$, giving us a subspace of dimension $\sum n_{x_i}$ which is contained in $\mathscr{H}$. Thus the space spanned by the functions (3.3) and (3.4) together contains a subspace of dimension n contained in $\mathscr{H}$. To see that there is an n-dimensional subspace lying entirely in $\mathscr{H}^\perp$, observe that $\mathscr{H}$ is spanned by the functions

$$\varphi_{N-k} \quad (k < n_\infty), \qquad \varphi_k \quad (k < n - n_\infty), \qquad \prod (x - x_i)^{n_{x_i}} x^k \quad (k < N - n).$$

Our $2n$ functions are all orthogonal to the last of these, whereas orthogonality to the remaining ones imposes n conditions, giving a subspace of dimension n which is contained in $\mathscr{H}^\perp$.

It follows from the preceding discussion that the space spanned by the functions (3.3) and (3.4) contains n linearly independent functions $\psi_1, \ldots, \psi_n$ lying in $\mathscr{H}$ and n linearly independent functions $\psi_{n+1}, \ldots, \psi_{2n}$ lying in $\mathscr{H}^\perp$. And we have a representation

$$[D, K] = \sum_{i,\, j=1}^{2n} A_{ij} \psi_i \otimes \psi_j \tag{3.5}$$

for some constants A_{ij} which can be determined from Eq. (3.2) once we have fixed the ψ_i. (We use the notation $a \otimes b$ for the operator with kernel $a(x)\, b(y)$.) It follows that also

$$[\varepsilon, K] = \sum_{i,\, j=1}^{2n} A_{ij} \varepsilon\psi_i \otimes \varepsilon\psi_j. \tag{3.6}$$

Here we used $\varepsilon D = D\varepsilon = I$, the antisymmetry of ε and the easy fact that $(a \otimes b)\, T = a \otimes (T'b)$ for any operator T. These will be used again below without comment.

The matrix $A = (A_{ij})$ is symmetric since K is symmetric and D is antisymmetric. (We hope this A will not be confused with the function A

appearing in Eq. (3.2).) Since K is the projection operator onto $\mathscr{H}$ the commutator $[D, K]$ takes $\mathscr{H}$ into $\mathscr{H}^{\perp}$ and $\mathscr{H}^{\perp}$ into $\mathscr{H}$. Hence

$$A_{ij}=0 \qquad \text{if} \quad i, j \leqslant n \quad \text{or} \quad i, j>n. \tag{3.7}$$

After a little more notation we shall be able to state the formulas. We already have the matrix A. We define the matrix B by

$$B_{ij}=(\varepsilon\psi_i, \psi_j).$$

Define J to be the matrix whose i, j entry equals 1 if $i=j\leqslant n$ and 0 otherwise. Finally, set

$$C=J+BA$$

and write A_0 for the matrix obtained from A by deleting its last n columns, C_0 for the matrix obtained from C by deleting its last n rows and C_{00} for the matrix obtained from C by deleting its last n rows and its last n columns. Observe that by Eq. (3.7) the first n rows of A_0 are zero.

Theorem 2. We have

$$S_N^{(4)}(x, y)=K_N(x, y)-\sum_{i>n,\, j}(A_0 C_{00}^{-1} C_0)_{ij}\,\psi_i(x)\,\varepsilon\psi_j(y) \tag{3.8}$$

$$S_N^{(1)}(x, y)=K_N(x, y)-\sum_{i\leqslant n,\, j}[AC(I-BAC)^{-1}]_{ji}\,\psi_i(x)\,\varepsilon\psi_j(y). \tag{3.9}$$

Proof. Using (3.5) we find

$$(I-K)\,DK\varepsilon=[D, K]\,K\varepsilon=\left(\sum_{i,j} A_{ij}\psi_i\otimes\psi_j\right)K\varepsilon=-\sum_{j\leqslant n,\, i} A_{ij}\psi_i\otimes\varepsilon\psi_j$$

since $K\psi_j=\psi_j$ when $j\leqslant n$ and $K\psi_j=0$ when $j>n$. Thus

$$I-(I-K)\,DK\varepsilon=I+\sum_{j\leqslant n,\, i} A_{ij}\psi_i\otimes\varepsilon\psi_j.$$

Now if we have a finite rank operator $\sum a_i\otimes b_i$ then

$$\left(I+\sum a_i\otimes b_i\right)^{-1}=I-\sum_{i,j} T_{ij}^{-1} a_i\otimes b_j \tag{3.10}$$

where T is the matrix with entries

$$T_{ij} = \delta_{ij} + (b_i, a_j).$$

In our case $i, j \leqslant n$ and

$$a_i = \sum_k A_{ki} \psi_k, \qquad b_i = \varepsilon \psi_i$$

so

$$T_{ij} = \delta_{ij} + \sum_k (\varepsilon \psi_i, \psi_k) A_{kj} = \delta_{ij} + \sum_k B_{ik} A_{kj}.$$

This equals $(I + BA)_{ij} = C_{ij}$ and so we have shown

$$(I - (I - K) DK\varepsilon)^{-1} = I - \sum_{i, j \leqslant n} (C_{00})^{-1}_{ij} \left(\sum_k A_{ki} \psi_k \otimes \varepsilon \psi_j \right)$$

whence

$$S^{(4)} = (I - (I - K) DK\varepsilon)^{-1} K = K - \sum_{i, j \leqslant n} \sum_k A_{ki} (C_{00})^{-1}_{ij} \psi_k \otimes K\varepsilon \psi_j. \quad (3.11)$$

To compute $K\varepsilon\psi_j$ we apply Eq. (3.6) to ψ_j, using the fact that $\psi_j \in \mathscr{H}$, to obtain

$$K\varepsilon\psi_j = \varepsilon\psi_j - \sum_{l, k} A_{lk} \varepsilon\psi_l (\varepsilon\psi_k, \psi_j) = \varepsilon\psi_j - \sum_{l, k} A_{lk} B_{kj} \varepsilon\psi_l$$

$$= \varepsilon\psi_j + \sum_l (BA)_{jl}\, \varepsilon\psi_l = \sum_l C_{jl}\, \varepsilon\psi_l. \quad (3.12)$$

Here we used the symmetry of A and the antisymmetry of B. Substituting this into (3.11) gives

$$S^{(4)} = (I - (I - K) DK\varepsilon)^{-1} K = K - \sum_{k, l} (A_0 C_{00}^{-1} C_0)_{kl}\, \psi_k \otimes \varepsilon \psi_l$$

which is the same as Eq. (3.8).

To derive Eq. (3.9) we use Eq. (3.6) and find that

$$(I - K)\, \varepsilon KD = [\varepsilon, K]\, KD = -\sum_{i, j} A_{ij} \varepsilon \psi_i \otimes DK\varepsilon\psi_j. \quad (3.13)$$

Using Eq. (3.5) again and the fact that $D\varepsilon = I$ we see that

$$DK\varepsilon\psi_j = K\psi_j + \sum_{k,\,l} A_{kl}\psi_k(\psi_l, \varepsilon\psi_j) = K\psi_j + \sum_{k,\,l} B_{jl}A_{lk}\psi_k.$$

Again we use the fact that $K\psi_j = \psi_j$ when $j \leqslant n$ and $K\psi_j = 0$ when $j > n$. If we recall the definitions of J and C we see that we have shown $DK\varepsilon\psi_j = \sum_k C_{jk}\psi_k$. Substituting this into Eq. (3.13) gives

$$(I-K)\,\varepsilon KD = -\sum_{i,\,j,\,k} A_{ij}C_{jk}\;\varepsilon\psi_i \otimes \psi_k$$

and so

$$I - (I-K)\,\varepsilon KD = I + \sum_{i,\,k} (AC)_{ik}\;\varepsilon\psi_i \otimes \psi_k.$$

We use Eq. (3.10) again, this time with $i, j \leqslant 2n$ and

$$a_i = \sum_k (AC)_{ki}\,\varepsilon\psi_k, \qquad b_i = \psi_i.$$

Now we have

$$T_{ij} = \delta_{ij} + \sum_k (\psi_i, \varepsilon\psi_k)(AC)_{kj} = \delta_{ij} - \sum_k B_{ik}(AC)_{kj} = (I - BAC)_{ij}.$$

Hence Eq. (3.10) gives

$$\begin{aligned}(I-(I-K)\,\varepsilon KD)^{-1} &= I - \sum_{i,\,j} (I-BAC)^{-1}_{ij} \sum_k (AC)_{ki}\,\varepsilon\psi_k \otimes \psi_j \\ &= I - \sum_{j,\,k} [AC(I-BAC)^{-1}]_{kj}\,\varepsilon\psi_k \otimes \psi_j.\end{aligned}$$

To obtain $S^{(1)\prime}$ we must right-multiply by K, which has the effect of imposing the restriction $j \leqslant n$. After taking transposes and changing notation we obtain Eq. (3.9).

Remark. Here is how to extend the results to the case where the second part of Eq. (2.4) may not be satisfied but the support $\mathscr{D}$ of w consists of a finite union of intervals. Denote now by x_i the poles of w'/w together with all finite end-points of $\mathscr{D}$ where w'/w is analytic. Then we can write

$$w(x) = \prod_i (x - x_i)^{\alpha_i}\, w_0(x)$$

where w_0 satisfies Eq. (2.4) and each $\alpha_i > -1$. Think of w, and therefore the kernels $K_N(x, y)$, $S_N^{(4)}(x, y)$ and $S_N^{(1)}(x, y)$, as functions of the α_i. Theorem 2 would apply to w itself if all the $\alpha_i > 0$ since then Eq. (2.4) would be satisfied. But the constituents of these kernels are real-analytic functions of the α_i, so the formulas for $\alpha_i > -1$ (and therefore for our given weight function w) can be obtained by analytic continuation of the formulas for $\alpha_i > 0$.

4. THE GAUSSIAN AND LAGUERRE ENSEMBLES

These are (essentially the only) cases where $n=1$ and are especially simple, as we shall now see.

By the symmetry of A and Eq. (3.7), A has the form

$$A = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$$

for some constant λ. There arise two functions, ψ_1 and ψ_2, the first lying in $\mathscr{H}$ and the second lying in $\mathscr{H}^\perp$.

Since the two 2×2 matrices A and B have 0 diagonal entries, AB is a diagonal matrix and therefore so is C. Therefore Eq. (3.12), in which $j=1$, says that $K\varepsilon\psi_1 = C_{11}\psi_1$. Since $K\varepsilon_{\mathscr{H}}$ is invertible $C_{11} \neq 0$, and so $\varepsilon\psi_1 \in \mathscr{H}$. This implies that all entries of B vanish, so $B=0$, $C=J$. It is immediate from these facts that

$$A_0 C_{00}^{-1} C_0 = AC(I - BAC)^{-1} = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}.$$

Hence by Theorem 2,

$$S_N^{(4)}(x, y) = K_N(x, y) - \lambda\psi_2(x)\, \varepsilon\psi_1(y),$$

$$S_N^{(1)}(x, y) = K_N(x, y) - \lambda\psi_1(x)\, \varepsilon\psi_2(y).$$

It remains to find the constant λ and the functions ψ_1 and ψ_2 in the two cases.

The Gaussian Ensembles. Here the pole is at $x = \infty$. Clearly $\psi_1 = \varphi_{N-1}$ and $\psi_2 = \varphi_N$ in this case. Moreover we have for this ensemble

$$a_N = \sqrt{N/2}, \qquad A(x) = -x, \qquad B(x) = C(x) = \sqrt{2N}$$

so we find from Eq. (3.2) that

$$A = \begin{pmatrix} 0 & -\sqrt{N/2} \\ -\sqrt{N/2} & 0 \end{pmatrix}$$

which gives $\lambda = -\sqrt{N/2}$. Therefore

$$S_N^{(4)}(x, y) = K_N(x, y) + \sqrt{N/2}\, \varphi_N(x)\, \varepsilon\varphi_{N-1}(y),$$
$$S_N^{(1)}(x, y) = K_N(x, y) + \sqrt{N/2}\, \varphi_{N-1}(x)\, \varepsilon\varphi_N(y).$$

The Laguerre Ensembles. Here $w(x) = x^\alpha e^{-x}$ and

$$p_j(x) = \sqrt{\frac{j!}{\Gamma(j+\alpha+1)}}\, L_j^{(\alpha)}(x) \tag{4.1}$$

where $L_j^{(\alpha)}$ is the generalized Laguerre polynomial. We assume at first that $\alpha > 0$ so that Eq. (2.4) holds. In the notation of Eq. (3.1) and (3.2) we have

$$a_N = -\sqrt{N(N+\alpha)},$$
$$A(x) = x^{-1}\left(N + \frac{\alpha}{2}\right) - \frac{1}{2},$$
$$B(x) = C(x) = -x^{-1}\sqrt{N(N+\alpha)}.$$

The pole is at $x = 0$, and if we set $\xi_j(x) = x^{-1}\varphi_j(x)$ then Eq. (3.2) becomes

$$\sqrt{N(N+\alpha)}\, (\xi_N(x)\ \xi_{N-1}(x)) \begin{pmatrix} -\sqrt{N(N+\alpha)} & N+\alpha/2 \\ N+\alpha/2 & -\sqrt{N(N+\alpha)} \end{pmatrix} \begin{pmatrix} \xi_N(y) \\ \xi_{N-1}(y) \end{pmatrix}. \tag{4.2}$$

Our functions ψ_1 and ψ_2 are linear combinations of ξ_N and ξ_{N-1}, with ψ_1 lying in $\mathscr{H}$ and ψ_2 lying in $\mathscr{H}^\perp$. Clearly ψ_1 is a constant times $p_{N-1}(0)\,\xi_N(x) - p_N(0)\,\xi_{N-1}(x)$. Since $L_N^{(\alpha)}(0)/L_{N-1}^{(\alpha)}(0) = (N+\alpha)/N$ we see using Eq. (4.1) that we may take

$$\psi_1 = \sqrt{N}\,\xi_N - \sqrt{N+\alpha}\,\xi_{N-1}.$$

For ψ_2, it follows from the discussion near the beginning of the last section that the appropriate linear combination of ξ_N and ξ_{N-1} may be found by requiring that it be orthogonal to φ_0. From the fact that

$\int_0^\infty L_m^{(\alpha)}(x)\, x^{\alpha-1} e^{-x}\, dx = \Gamma(\alpha)$ and from Eq. (4.1) we see that the linear combination

$$\psi_2 = \sqrt{N+\alpha}\, \xi_N - \sqrt{N}\, \xi_{N-1}$$

does the job.

Solving for ξ_N and ξ_{N-1} in terms of ψ_1 and ψ_2 and substituting into Eq. (4.2) we obtain for the kernel of $[D, K]$,

$$-\frac{\sqrt{N(N+\alpha)}}{2} (\psi_1(x)\, \psi_2(x)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1(y) \\ \psi_2(y) \end{pmatrix}.$$

Therefore

$$\lambda = -\frac{\sqrt{N(N+\alpha)}}{2}.$$

Hence for this ensemble we find that $S_N^{(4)}(x, y)$ is equal to $K_N(x, y)$ plus

$$\frac{\sqrt{N(N+\alpha)}}{2} (\sqrt{N+\alpha}\, \xi_N(x) - \sqrt{N}\, \xi_{N-1}(x)) \times (\sqrt{N}\, \varepsilon\xi_N(y) - \sqrt{N+\alpha}\, \varepsilon\xi_{N-1}(y)) \tag{4.3}$$

and that $S_N^{(1)}(x, y)$ is equal to $K_N(x, y)$ plus

$$\frac{\sqrt{N(N+\alpha)}}{2} (\sqrt{N}\, \xi_N(x) - \sqrt{N+\alpha}\, \xi_{N-1}(x)) \times (\sqrt{N+\alpha}\, \varepsilon\xi_N(y) - \sqrt{N}\, \varepsilon\xi_{N-1}(y)). \tag{4.4}$$

These were established for $\alpha > 0$. For $-1 < \alpha \leqslant 0$ we must find the analytic continuations of the factors in Eq. (4.3) and (4.4). The first factors cause no difficulty since they are defined and analytic for $\alpha > -1$. The same is true of the second factor in Eq. (4.3) since $\sqrt{N}\, p_N - \sqrt{N+\alpha}\, p_{N-1}$ has zero constant term for $\alpha > 0$ (since $\psi_1 \in \mathscr{H}$) and so for all α.

The second factor in Eq. (4.4) requires analytic continuation. Assuming at first that $\alpha > 0$ we write it as

$$-\int_y^\infty (\sqrt{N+\alpha}\, \xi_N(z) - \sqrt{N}\, \xi_{N-1}(z)\, dz + \tfrac{1}{2} \int_0^\infty (\sqrt{N+\alpha}\, \xi_N(z) - \sqrt{N}\, \xi_{N-1}(z))\, dz. \tag{4.5}$$

Now the fact $\varepsilon\psi_1 \in \mathscr{H}$ established earlier implies that $\int_0^\infty \psi_1(z)\,dz = 0$. This is equivalent to

$$\sqrt{N}\int_0^\infty \xi_N(z)\,dz = \sqrt{N+\alpha}\int_0^\infty \xi_{N-1}(z)\,dz$$

so the last integral, with its factor 1/2, is equal to

$$\frac{1}{2}\int_0^\infty \left(\sqrt{N+\alpha} - \frac{N}{\sqrt{N+\alpha}}\right)\xi_N(z)\,dz$$

$$= \frac{\alpha}{2\sqrt{N+\alpha}}\int_0^\infty z^{\alpha/2-1}e^{-z/2}p_N(z)\,dz = -\frac{1}{\sqrt{N+\alpha}}\int_0^\infty z^{\alpha/2}(e^{-z/2}p_N(z)'\,dz.$$

Hence the second factor in Eq. (4.4) is equal to

$$-\int_y^\infty (\sqrt{N+\alpha}\,\xi_N(z) - \sqrt{N}\,\xi_{N-1}(z))\,dz$$
$$-\frac{1}{\sqrt{N+\alpha}}\int_0^\infty z^{\alpha/2}(e^{-z/2}p_N(z))'\,dz. \tag{4.6}$$

This is analytic for all $\alpha > -1$ and so provides the desired analytic continuation.

It is to be remarked (we thank the referee for this observation) that the second integral in Eq. (4.5), which is equal to $(N!/\Gamma(N+\alpha))^{1/2}$ $\int_0^\infty z^{\alpha/2-1}e^{-z/2}L_N^{(\alpha-1)}\,dz$, has already occurred in the study of the Laguerre ensemble and can be explicitly evaluated in terms of gamma functions[5, 11].

When $\alpha = 0$

$$L_N(x) - L_{N-1}(x) = \frac{x}{N}L_N'(x),$$

and we find that

$$S_N^{(4)}(x, y) = K_N(x, y) + \tfrac{1}{2}e^{-x/2}L_N'(x)\int_0^y e^{-z/2}L_N'(z)\,dz,$$

$$S_N^{(1)}(x, y) = K_N(x, y) + \tfrac{1}{2}e^{-x/2}L_N'(x)\left(\int_0^y e^{-z/2}L_N'(z)\,dz + 1\right).$$

For both we used the fact $\int_0^\infty \psi_1(z)\,dz = 0$ once again and, for the latter, Eq. (4.6). These formulas, in somewhat different form, are those obtained earlier using skew-orthogonal polynomials.[6, 7]

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